

# 10/2/24 Lecture 9: PAC Learning Cont'd

Contents:

- 1). Noise stability  $\rightarrow$  Fourier concentration
- 2). Stein's lemma (baby version)

Let  $f: \{\pm 1\}^n \rightarrow \{\pm 1\}$ ,  $\gamma \in (0, 1/2)$

Recall noise sensitivity

$$NS_\gamma(f) \triangleq \Pr_{x, x'}[f(x) \neq f(x')],$$

where  $x \sim \{\pm 1\}^n$  and  $x'$  given by flipping each coordinate of  $x$  with probability  $\gamma$ .

Lemma:  $NS_\gamma(f) = \frac{1}{2} - \frac{1}{2} \sum_{S \subseteq [n]} (1-2\gamma)^{|S|} \hat{f}[S]^2.$

Pf: Note  $\mathbb{E}[f(x)f(y)] = \Pr[f(x)=f(y)] - \Pr[f(x) \neq f(y)]$   
 $= 1 - 2 \Pr[f(x) \neq f(y)]$   
 $= 1 - 2NS_\gamma(f),$

so  $NS_\gamma(f) = \frac{1}{2} - \frac{1}{2} \mathbb{E}[f(x)f(y)]$   
 $= \frac{1}{2} - \frac{1}{2} \sum_{S, T} \hat{f}[S] \hat{f}[T] \mathbb{E}[x_S y_T]$

Note:  $\mathbb{E}[x_S y_T] = \underbrace{\mathbb{E}[x_{S \cap T}] \mathbb{E}[y_{T \setminus S}]}_{= 0 \text{ if } S \neq T} \cdot \mathbb{E}[x_{S \setminus T} y_{S \setminus T}]$

and  $\mathbb{E}(x_i y_i) = (1-\gamma) - \gamma = 1-2\gamma$ ,  
 so  $\mathbb{E}(x_S y_T) = \mathbb{1}(S=T) \cdot (1-2\gamma)^{|S|}$ . □

Lemma: 
$$\sum_{|S| \geq 1/\gamma} \hat{f}(S)^2 \leq NS_\gamma(f)$$

Pf: 
$$2NS_\gamma(f) = 1 - \sum_{S \subseteq [n]} (1-2\gamma)^{|S|} \hat{f}(S)^2$$

by previous lemma.

Note  $1 = \mathbb{E}(f^2) = \sum_{S \subseteq [n]} \hat{f}(S)^2$ , so

$$\rightarrow = \sum_{S \subseteq [n]} \hat{f}(S)^2 (1 - (1-2\gamma)^{|S|})$$

$$\geq \sum_{|S| \geq 1/\gamma} \hat{f}(S)^2 \underbrace{(1 - (1-2\gamma)^{|S|})}_{\geq 1 - e^{-2}}$$

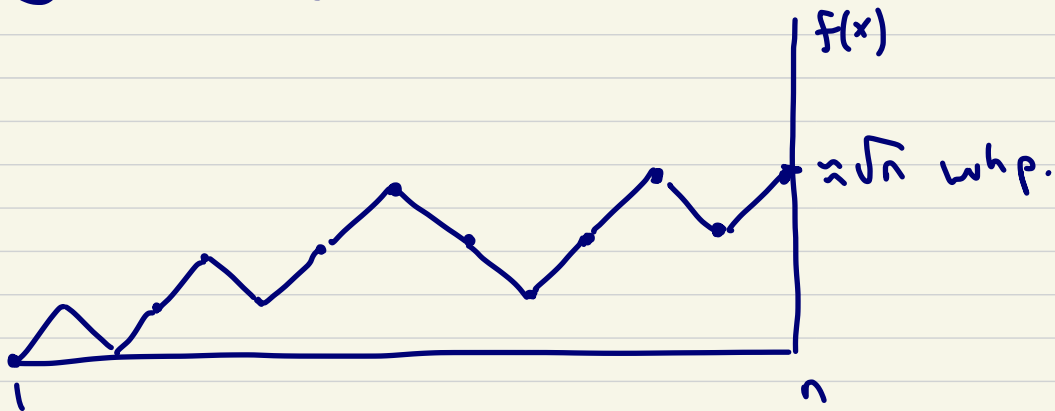
so rearranging gives desired bound. □

Thm (Peres '04): If  $f(x) = \text{sgn}(\langle w, x \rangle)$ , then

$$NS_\gamma(f) \leq \sqrt{\gamma}.$$

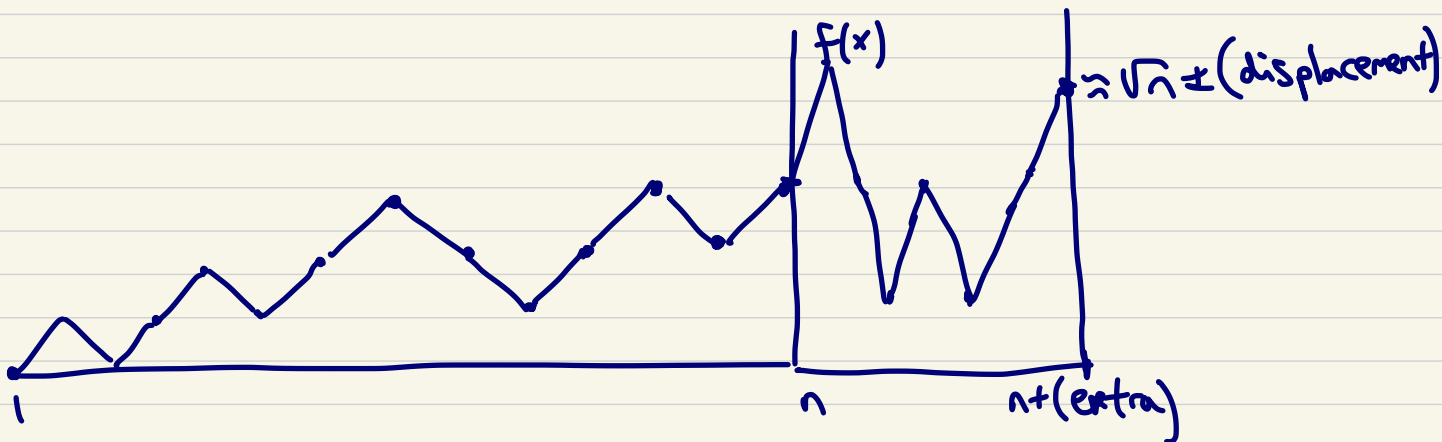
Pf sketch for baby version where  $w = (1, \dots, 1)$ ,  
 i.e. when  $f$  is Majority  $(x_1, \dots, x_n)$ :

Can view  $f(x)$  for  $x \sim \{\pm 1\}^n$  as random walk:



sort of

Can view  $f(y)$  as Continuation of this random walk  
 for  $\approx \gamma n$  steps, where each step we move by  
 $2$  instead of  $1$  (b/c we are flipping coordinates from  
 $\pm 1$  to  $\mp 1$ ). so



in order for  $f(y) \neq f(x)$ , displacement must exceed  $\sqrt{n}$ . But

$$\Pr[\text{displacement} > \sqrt{n}] \leq \frac{\mathbb{E}(\text{displacement})}{\sqrt{n}} \\ \leq \sqrt{\gamma}. \quad \square$$

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Corollary: Let  $h: \{\pm 1\}^k \rightarrow \{\pm 1\}$ ,  $w_1, \dots, w_k \in \mathbb{R}^n$ .

Define  $f(x) \triangleq h(\text{sgn}(\langle w_1, x \rangle), \dots, \text{sgn}(\langle w_k, x \rangle))$ .

Then

$$NS_{\gamma}(f) \leq k\sqrt{\gamma}$$

$$\begin{aligned} \text{PF: } \Pr[f(x) \neq f(y)] &\leq \sum_{i=1}^k \Pr[\text{sgn}(\langle w_i, x \rangle) \neq \text{sgn}(\langle w_i, y \rangle)] \\ &= k \cdot \max_w NS_{\gamma}(\langle w, x \rangle) \\ &\leq k\sqrt{\gamma}. \quad \square \end{aligned}$$

For certain  $h$ , can do much better, e.g.

[Kane '14]: If  $f = \text{AND}(\text{sgn}(\langle w_1, x \rangle), \dots, \text{sgn}(\langle w_k, x \rangle))$ ,

then  $\text{NS}_\gamma(f) = O(\sqrt{\lg(k) \cdot \gamma})$ , so

intersections of halfspaces can be learned in time

$$n^{O(\lg(k)/\epsilon^2)}$$

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Stein's lemma (baby version):

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be sufficiently "regular". Then for  $x \sim N(0, 1)$ ,

$$\textcircled{1} \quad \mathbb{E}[f(x) \cdot x] = \mathbb{E}[f'(x)].$$

$$\textcircled{2} \quad \mathbb{E}[f(x) \cdot (x^2 - 1)] = \mathbb{E}[f''(x)]. \quad \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}$$

Pf: Let  $\gamma$  denote Gaussian density. Note:

$$x \cdot \gamma(x) = -\gamma'(x)$$

$$\int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{int. by parts}} \cdot \underbrace{x \cdot \gamma(x)}_{\text{int. by parts}} dx = \int_{-\infty}^{\infty} f'(x) \cdot \gamma(x) dx + \underbrace{f(x) \gamma(x)}_{\substack{= 0 \\ \text{b/c } \gamma \rightarrow 0}} \Big|_{-\infty}^{\infty}$$

$$= \oplus [f'(x)]$$

Similarly, note:

$$\gamma''(x) = (-x \cdot \gamma(x))' = (x^2 - 1)\gamma(x)$$

$$\int_{-\infty}^{\infty} f(x) \cdot (x^2 - 1)\gamma(x) dx = \int_{-\infty}^{\infty} f'(x) \cdot (x \cdot \gamma(x)) dx + \underbrace{f(x) \cdot x \cdot \gamma(x)}_{=0} \Big|_{-\infty}^{\infty}$$

$$= \int_{-\infty}^{\infty} f''(x) \gamma(x) dx + \underbrace{f'(x) \cdot \gamma(x) dx}_{=0} \Big|_{-\infty}^{\infty}$$

$$\rightarrow \oplus [f''(x)]. \quad \square$$