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# Lecture 4: SOS Basics

<u>Claim</u>: For any univariate polynomial $p$ s.t. $p(x) \geq 0$ for all $x \in \mathbb{R}$, $p$ can be written as a sum of squares.

<u>Pf</u>: $\ell := \deg(p)$, induct on $\ell$.

Base case of $\ell = 0$ trivial.
For $\ell > 0$, let $z \geq 0$ be min value of $p$, achieved at some $x = a$, and consider

$$p(x) - z = (x-a)^t q(x) \quad \text{for } q(a) \neq 0.$$

Order of $t$ must be even (why?)
so $q(x) \geq 0 \;\; \forall\, x \in \mathbb{R}$ and $\deg(q) < \ell$, completing inductive step. $\square$

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Claim is not generally true for polynomials in more than one variable.

[Hilbert '88]: nonconstructive proof

[Motzkin '67]: explicit example:

$$p(x,y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

- $p(x,y) \geq 0 \;\forall x,y$ by AM-GM inequality

- Theorem: $p(x,y)$ not a sum-of-squares

Pf sketch: If it were SOS, would have to consist of squares of the form

$$(ax^2y + bxy^2 + cxy + 1)^2,$$

but the coefficient of $x^2y^2$ is non-negative for such polynomials. $\square$

(note: can also prove theorem with an SDP solver...)

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SOS Proof given constraints:

Given: Variables: $\vec{x} = (x_1, \ldots, x_n)$

Constraints: $p_1(\vec{x}) = 0, \ldots, p_r(\vec{x}) = 0$

$q_1(\vec{x}) \geq 0, \ldots, q_s(\vec{x}) \geq 0$

for polynomials $p_1, \ldots, p_r, q_1, \ldots, q_s$ of degree $\leq t$.

There is a deg-$t$ SOS proof of the polynomial inequality $f(\vec{x}) \geq 0$ if one can write

$$f(\vec{x}) = \sum_{S \subseteq [s]} \prod_{i \in S} q_i(\vec{x}) \cdot SOS_S(\vec{x})$$

$$+$$

$$\sum_{T \subseteq [t]} \prod_{i \in T} q_i(\vec{x}) \cdot g_T(\vec{x})$$

$$+$$

$$SOS'(\vec{x})$$

where $SOS_S(\vec{x})$ and $SOS'(\vec{x})$ are sum-of-squares polynomials, $g_T(\vec{x})$ are arbitrary polynomials, and all expressions above are polynomials of degree at most $t$.