12/4

Lecture 25 : Diffusions and GMMs

Thm (Jackson): For $f : \mathbb{R} \to \mathbb{R}$ which is L-Lipschitz, for any $\ell \geq 1$ there exists polynomial p of degree-ℓ s.t.

$$\sup_{|x| \leq r} |f(x) - p(x)| \lesssim rL/\ell$$

Proof outline: ① Prove for trigonometric polynomials

ⓐ Prove for $f(x) \triangleq x$ for $x \in [-\pi, \pi)$

ⓑ Prove for general f over $[-\pi, \pi)$

② Conclude for (algebraic polynomials)

1a: Trigonometric polynomial of degree n:

$$T_n \triangleq \left\{ a_0 + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) \,,\ a_k, b_k \in \mathbb{R} \right\}$$

Lem: Given distinct points $x_0, \ldots, x_{2n}$ in $[-\pi, \pi)$ and values $y_1, \ldots, y_{2n}$, there is unique interpolator $p \in T_n$ satisfying $p(x_i) = y_i \ \forall i$.

Pf. Enough to show any nonzero $p \in T_n$ has at most $2n$ zeros. Suppose it has $2n+1$ zeros $x_1, \ldots, x_{2n+1}$.

Then write $p(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} c_k z^k$

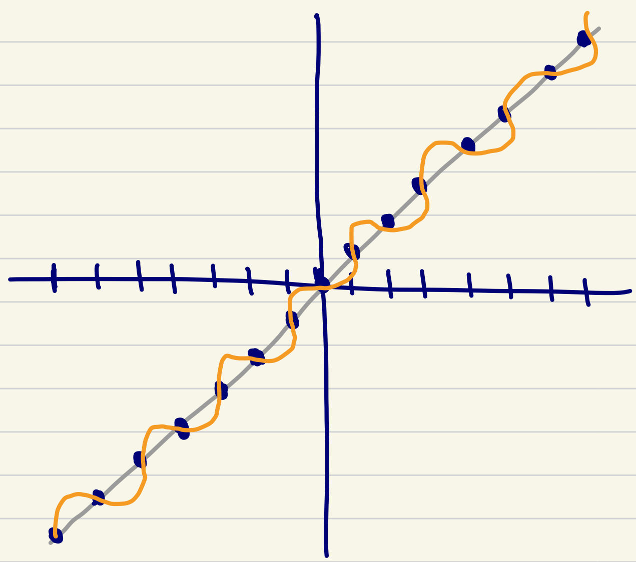
for $z = e^{ix}$. Then $z^n p(x) = \sum_{k=0}^{2n} c_k z^k$

is a deg-$2n$ nonzero polynomial vanishing at $2n+1$ distinct values of z, a contradiction. $\square$

<u>Lem</u>: $\exists\, p \in T_n$ s.t. for $f(x) \stackrel{\circ}{=} x$ for $x \in [-\pi, \pi)$,

$$\sup_{x \in [-\pi, \pi)} |p(x) - f(x)| \lesssim \frac{1}{n+1}$$

<u>Pf</u>: Let p be the unique interpolator for



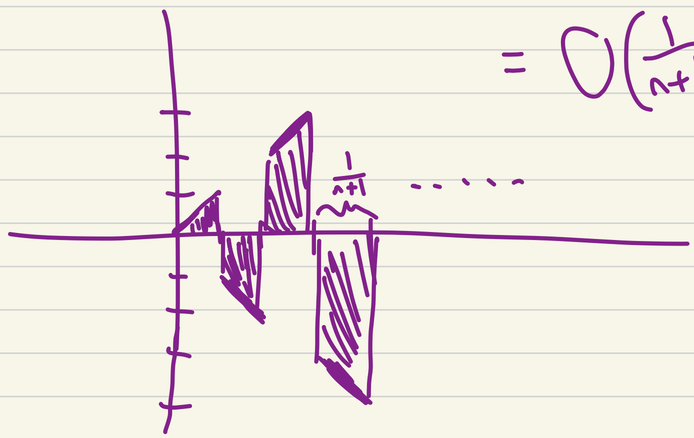
at the nodes $\left\{ \frac{k}{n+1}\pi : k = -n, -n+1, \dots, n-1, n \right\}$

Note that because $p(x) = -p(-x)$ for all nodes, p must be of the form $\sum_{k=1}^{n} a_k \sin(kx)$.

Furthermore, $\text{sign}(p - f)$ oscillates between the nodes, so

$$\int_{-\pi}^{\pi} |f(x) - p(x)| = \Bigg| \underbrace{\int_{-\pi}^{\pi} f(x)\, s(x)\, dx}_{\parallel} - \underbrace{\int_{-\pi}^{\pi} p(x)\, s(x)\, dx}_{=0 \text{ by symmetry}} \Bigg|$$

$$\underbrace{2\int_{0}^{\pi} x \cdot s(x)}_{= O\left(\frac{1}{n+1}\right)}$$



$\square$

(1b) : By integration by parts, given L-Lipschitz, periodic f,

$$\int_{-\pi}^{\pi} z\, f'(z + x + \pi)\, dz$$

$$= \underbrace{-\int_{-\pi}^{\pi} f(z+x+\pi)\, dz}_{= \text{ constant } C \text{ by periodicity!}} + \underbrace{\Big(z f(z+x+\pi)\Big)\Big|_{z=-\pi}^{z=\pi}}_{2\pi f(x)}$$

so $\quad 2\pi f(x) = C + \displaystyle\int_{-\pi}^{\pi} z\, f'(z + x + \pi)\, dz$

$\uparrow$ can approximate using

trigonometric polynomial p from part (1a)

If z replaced w/ approximation p, we get approx error

$$\int_{-\pi}^{\pi} (z-p)\, f'(z+x+\pi)\, dz$$

(Cauchy-Schwarz)

$$\leq \underbrace{\int_{-\pi}^{\pi} |z-p|\, dz}_{\lesssim \frac{1}{n+1}} \cdot \underbrace{\max_{\hat{x}} |f'(\hat{x})|}_{\leq L \text{ by assumption}}$$

$$\lesssim \frac{L}{n+1}$$

$\square$

---

(2) Can convert b/t (algebraic) polynomials q and <u>even</u> trigonometric polynomials p via

$$p(x) = q(\cos x)$$

If $g: [-1,1] \to \mathbb{R}$ is L-Lipschitz, then $x \mapsto g(\cos x)$ is L-Lipschitz periodic function over $[-\pi, \pi)$, So exists trig. poly p s.t.

$$|p(x) - g(\cos x)| \lesssim \frac{L}{n+1} \quad \forall x.$$

Consider $\hat{p}(x) = \dfrac{p(x) + p(-x)}{2}$. We also have

$$|\hat{p}(x) - g(\cos x)| \lesssim \frac{L}{n+1} \quad \forall x$$

Define $q(y) \triangleq \hat{p}(\arccos y)$

(well-defined b/c $\hat{p}$ even). $\square$