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Lecture 21: Stochastic calculus, Langevin

Ornstein-Uhlenbeck process:

$$dX_t = -X_t \, dt + \sqrt{2} \, dB_t$$

$$Y_t \triangleq X_t \cdot e^t$$

$$\Rightarrow dY_t = e^t dX_t + e^t X_t d_t \qquad \text{(product rule for Ito integrals)}$$

$$= e^t(-X_t dt + \sqrt{2} dB_t) + e^t X_t d_t$$

$$= \sqrt{2} e^t dB_t$$

$$\Rightarrow Y_t \sim N\left(Y_0, \int_0^t (\sqrt{2} e^s)^2 ds\right)$$

$$= N(Y_0, (e^{2t} - 1))$$

$$\Rightarrow X_t = e^{-t} Y_t \sim N\left(X_0 \cdot e^{-t}, \sqrt{1 - e^{-2t}}\right)$$

as $t \to \infty$, $X_t \to N(0,1)$

Ito's lemma:

$$X_{t+h} \simeq X_t + h \cdot \mu_t(X_t) + \sqrt{h} \, \sigma_t(X_t) \, g$$

$$f(X_{t+h}) \simeq f(X_t) + \left\langle \nabla f(X_t), h \cdot \mu_t(X_t) + \sqrt{h} \, \sigma_t(X_t) g \right\rangle$$

$$+ \frac{1}{2}\left\langle \nabla^2 f(x), h^2 \mu_t(X_t)^{\otimes 2} + h \, \sigma_t(X_t) g g^T \sigma_t(X_t)^T + h^{3/2} \mu_t(X_t) g^T \sigma_t(X_t) + h^{3/2} \sigma_t(X_t) g \mu_t(X_t)^T \right\rangle + \cdots$$

Kolmogorov backward:

$$\partial_t P_t f = \partial_t \lim_{h\to 0} \frac{P_{t+h}f - P_t f}{h}$$

$$= \partial_t P_t \underbrace{\lim_{h\to 0} \frac{P_h f - f}{h}}_{\mathcal{L} f} \qquad = \partial_t \lim_{h\to 0} \frac{P_h f - f}{h} P_t$$

$$= \partial_t P_t \mathcal{L} f \qquad = \partial_t \mathcal{L} P_t f$$

Kolmogorov forward:

$$\mathbb{E}[f(x_t)] = \int P_t f \, dq_0 = \int f \, d P_t^* q_0 \text{ (← adjoint of } P_t\text{)}, \qquad \text{so } q_t = P_t^* q_0$$

$$\partial_t \mathbb{E}[f(x_t)] = \partial_t \int P_t f \, dq_0$$

$$= \int P_t \mathcal{L} f \, dq_0$$

$$= \int f \, d (P_t \mathcal{L})^* q_0$$

$$= \int f \, d\mathcal{L}^* q_t$$

so $\boxed{\partial_t q_t = \mathcal{L}^* q_t}$

Derivation of $\mathcal{L}^*$:

$$\int f \mathcal{L}^* \, dq_t = \int \mathcal{L} f \, dq_t$$

$$= \int \left(\langle \nabla f, \mu_t \rangle + \frac{1}{2} \langle \nabla^2 f, \sigma_t \sigma_t^T \rangle \right) dq_t$$

$$= -\int f \cdot \mathrm{div}(\mu_t q_t) \, dx - \frac{1}{2} \int \langle \nabla f, \nabla \cdot \sigma_t \sigma_t^T q_t \rangle \, dx$$

$$= -\int f \cdot \mathrm{div}(\mu_t q_t) \, dx + \frac{1}{2} \int f \cdot \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (\sigma_t \sigma_t^T q_t) \, dx$$

so

$$\boxed{\partial_t q_t = -\mathrm{div}(\mu_t \cdot q_t) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (\sigma_t \sigma_t^T \cdot q_t)}$$

<u>Fokker-Planck</u>

<u>Integration by parts for Dirichlet form:</u>

$$-\int f(x) \, \mathcal{L} g(x) \, q(x) \, dx$$

$$= -\int f(x) \left(\Delta g(x) - \langle \nabla V(x), \nabla g(x) \rangle \right) q(x) \, dx$$

$$= \int \langle \nabla g(x), \underbrace{\nabla (f(x) \cdot q(x))}_{=} \rangle \, dx + \boxed{\int f(x) \langle \nabla V(x), \nabla g(x) \rangle q(x) \, dx}$$

$$\nabla f(x) \cdot q(x) + \nabla q(x) \cdot f(x)$$

$$= \nabla f(x) \cdot q(x) \boxed{- q \nabla V(x) \cdot f(x)}$$

$$= \int \langle \nabla g(x), \nabla f(x) \rangle q(x)\, dx \quad \square$$

Poincaré $\Leftrightarrow$ Mixing in χ^2:

$$\chi^2(p\|q) = \int \left(\frac{p(x)}{q(x)} - 1 \right)^2 dq(x)$$

$$\partial_t \chi^2(q_t \| q) = 2\int \left(\frac{q_t(x)}{q(x)} - 1 \right) \cdot \frac{\overbrace{\partial_t q_t(x)}^{= \mathcal{L}^* q_t(x)}}{q(x)} \, dq(x)$$

$$= -2\, \mathcal{E}\left(\frac{q_t}{q}, \frac{q_t}{q} \right)$$

$$\geq -2/C_p(q) \cdot \underbrace{\mathrm{Var}_q\left(\frac{q_t}{q} \right)}_{\chi^2(q_t\|q)}$$

(Gronwall's inequality) $\Longrightarrow$ $$\chi^2(q_t\|q) \leq \exp\left(-\frac{2t}{C_p(q)} \right) \cdot \chi^2(q_0\|q).$$