DECONVOLUTION

$$\hat{q}[\omega] = \frac{1}{k}\sum_{j=1}^{k} \hat{q}_i[\omega]$$

$$= \frac{1}{k}\sum_{j=1}^{k} \hat{K}_{\mu_j}[\omega]$$

(translation)
$$= \frac{1}{k}\sum_{j=1}^{k} \hat{K}[\omega] \cdot e^{2\pi i \cdot \langle \omega, \mu_j \rangle},$$

so for any ω s.t. $\hat{K}(\omega) > 0$,

$$\frac{1}{k}\sum_{j=1}^{k} e^{2\pi i \cdot \langle \omega, \mu_j \rangle} = \frac{\boxed{\hat{q}(\omega)}}{\boxed{\hat{K}(\omega)}}$$

Known quantity

Can estimate from samples:

$$\hat{q}[\omega] = \int_{-\infty}^{\infty} q(x)\, e^{-2\pi i \langle \omega, x \rangle}\, dx$$

$$= \mathbb{E}_{x \sim q}\left[e^{-2\pi i \langle \omega, x \rangle}\right]$$

$$\simeq \frac{1}{n}\sum_{j=1}^{n} e^{-2\pi i \langle \omega, x_j \rangle}$$

Matrix Pencil Method

[Hua - Sarkar '90]:

Given: query access to

$$\hat{G} : \omega \mapsto \frac{1}{k} \sum_j e^{2\pi i \, \omega \cdot \mu_j} + \text{noise}$$

for any $|\omega| \leq \frac{1}{\pi\sigma}$.

Henceforth, assume WLOG that $\sigma = \frac{1}{\pi}$.

$\boxed{\text{For now, let's pretend noise} = 0}$

We will query $\hat{G}$ on a grid. Let $m \in \mathbb{N}$ be a parameter.
Define $\alpha_\ell := \hat{G}\left[\ell \cdot \frac{1}{2m}\right]$ and form the matrix

$$A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} \\ \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_m \\ \alpha_2 & \vdots & \ddots & \cdots & \alpha_{m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1} & \alpha_m & \alpha_{m+1} & \cdots & \alpha_{2m-2} \end{pmatrix}$$

("Hankel matrices")

$$B = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_m \\ \alpha_2 & \alpha_3 & \cdots & \cdots & \alpha_{m+1} \\ \alpha_3 & \vdots & \ddots & \cdots & \alpha_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_m & \alpha_{m+1} & \alpha_{m+2} & \cdots & \alpha_{2m-1} \end{pmatrix}$$

Obs

$$A = \frac{1}{k} \cdot \overset{m\times k}{V} V^T$$

$$B = \frac{1}{k} V \cdot \operatorname{diag}\Big(\overbrace{e^{2\pi i \mu_1/2m}}^{z_1}, \dots, \overbrace{e^{2\pi i \mu_k/2m}}^{z_k}\Big) \cdot V^T$$

where V is the <u>Vandermonde matrix</u>

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_k \\ z_1^2 & z_2^2 & \cdots & z_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_k^{m-1} \end{pmatrix}$$

Pf: For any $0 \le r, s < m$,

$$\begin{aligned} A_{r,s} &= \alpha_{r+s} \\ &= \frac{1}{k} \sum_j e^{2\pi i (r+s)\mu_j/2m} \\ &= \frac{1}{k} \sum_j \underbrace{e^{2\pi i r \cdot \mu_j/2m}}_{z_j^r = V_{r,j}} \cdot \underbrace{e^{2\pi i s \cdot \mu_j/2m}}_{z_j^s = V_{s,j}} \\ &= \frac{1}{k} \sum_j V_{r,j} V_{s,j} \\ &= \frac{1}{k} (VV^T)_{r,s} \end{aligned}$$

$$B_{r,s} = \alpha_{r+s+1}$$
$$= \frac{1}{k} \sum_j V_{r,j} \cdot V_{s,j} \cdot \underbrace{e^{2\pi i \cdot \mu_j / 2m}}_{z_j}$$
$$= \frac{1}{k} \left(V \operatorname{diag}(z_1, \ldots, z_k) V^T \right)_{r,s}$$

Obs 1 ensures A, B obey the following nice relation.

Denote $D = \operatorname{diag}(z_1, \ldots, z_k)$

Then

pseudo-inverse

$$AB^+ = VV^T (VDV^T)^+$$
$$= VV^T (V^T)^+ D^{-1} V^+$$
$$= V D^{-1} V^+,$$

so $\boxed{\text{eigenvalues of } AB^+ \text{ tell us } z_1^{-1}, z_2^{-1}, \ldots, z_k^{-1}.}$

Recall $z_j^{-1} = e^{-2\pi i \cdot \mu_j / 2m}$,

So we can recover μ_j's from computing the modulus of z_j^{-1}'s.*

That was for noise $= 0$.

What about the noisy case?

As we'll see in pset 1, robustness of simultaneous diagonalization hinges on how close V is to being degenerate.

Need to lower bound

$$\sigma_{min}(V)$$

$$\underset{\text{min. singular value}}{\uparrow}$$

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* Technically only up to a period, e.g. $e^{-2\pi i \, \mu_j / 2m} = e^{-2\pi i \, (\mu_j + 2m)/2m}$, but this can be handled by varying m

Bounding $\sigma_{min}(V)$:

(1) when k is small (i.e. "no diffraction limit"):

Let $m=k$, so $z_j = e^{2\pi i \cdot \mu_j / 2k}$

Let $\sigma_1 \le \cdots \le \sigma_k$ be singular values of V

$$\prod_{j=1}^{k} \sigma_j = |\det(V)| = \prod_{1 \le i < j \le k} |z_i - z_j|$$

$$= \prod \left| e^{2\pi i \, \mu_i / 2k} - e^{2\pi i \, \mu_j / 2k} \right|$$

$$= \prod \left| e^{2\pi i (\mu_i - \mu_j)/2k} - 1 \right|$$

$$\ge \left| e^{2\pi i \cdot \Delta / 2k} - 1 \right|^{\binom{k}{2}}$$

$$\approx \Omega(\Delta / k)^{\binom{k}{2}}$$

easy to show $\sigma_j \le k \quad \forall j$.

So $\sigma_{min}(V) = \sigma_1 \ge \dfrac{\Omega(\Delta/k)^{\binom{k}{2}}}{k^{k-1}}$ $\quad (*)$

For $k = O(1)$, $(*) = \text{poly}(1/\Delta)$

② When k is large,

For any $\lambda \in \mathbb{S}^{k-1}$, w.t.s. $\|V\lambda\|$ not small

$$\|V\lambda\|^2 = \sum_{l=0}^{m-1} \left| \sum_{j=1}^{k} \lambda_j \, e^{2\pi i \cdot \mu_j \cdot \frac{l}{2m}} \right|^2$$

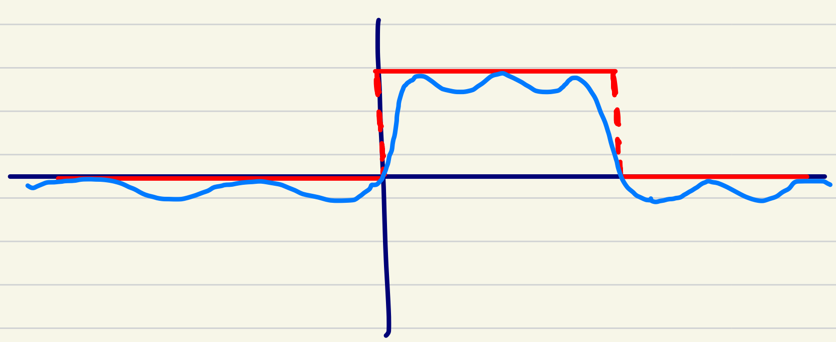
(by symmetry)

$$\geq \frac{1}{2} \sum_{l=-m+1}^{m-1} \left| \sum_{j=1}^{k} \lambda_j \, e^{2\pi i \cdot \mu_j \cdot \frac{l}{2m}} \right|^2$$

This step can be made rigorous either using Poisson summation or matrix Chernoff $\longrightarrow$

$$\text{"="} \; \frac{1}{2} \int_{-\infty}^{\infty} \mathbb{1}\left[|y| \leq \frac{m-1}{2m}\right] \cdot \left| \sum_{j=1}^{k} \lambda_j \, e^{2\pi i \cdot \mu_j \cdot y} \right|^2 dy$$

We will lower bound $\mathbb{1}[\cdots]$ by a special function $B(y)$



$$\geq \frac{1}{2}\int_{-\infty}^{\infty} B(y) \sum_{j,j'} \lambda_j \lambda_{j'} e^{2\pi i(\mu_j - \mu_{j'})y} \, dy$$

$$= \frac{1}{2}\sum_{j,j'} \lambda_j \lambda_{j'} \int_{-\infty}^{\infty} B(y)\, e^{2\pi i(\mu_j - \mu_{j'})y} \, dy$$

$$= \frac{1}{2}\sum_{j,j'} \lambda_j \lambda_{j'} \, \hat{B}[\mu_j - \mu_{j'}] \qquad (\varnothing)$$

Theorem [Beurling] (useful esp. in analytic number theory)

For any $\tau > 0$, exists $B : \mathbb{R} \to \mathbb{R}$ s.t.

1). [minorizes] $B(y) \leq \mathbb{1}[|y| \leq \tau/2]$

2). [Compact Fourier] $\hat{B}[w] = 0$ if $|w| > \frac{1}{\tau}$

3) [nontrivial] $\hat{B}[0] > 0$

Note if we take this B for

$\tau = \frac{m-1}{m} \approx 1$, then if $\Delta > 1$,

$\hat{B}[\mu_j - \mu_{j'}] = 0 \quad \forall \; j \neq j'$!

So $(\varnothing) = \frac{1}{2}\sum_j \lambda_j^2 \cdot \hat{B}[0]$,

i.e. $\sigma_{\min}(V) \geq \Omega(1)$. $\square$