Lecture 19 : Approximate Message Passing

(See slides for derivation of AMP, state evolution)

If we set $f_t(y) = \mathbb{E}[X \mid \mu_t X + \sigma_t Z = y]$,

then $\underset{X,Z}{\mathbb{E}}\left[X f_t(\mu_t X + \sigma_t Z)\right] = \underset{X,Z}{\mathbb{E}}\left[f_t(\mu_t X + \sigma_t Z)^2\right]$

$$= \mathbb{E}\left[\mathbb{E}[X \mid \mu_t X + \sigma_t Z]^2\right] = \sigma_{t+1}^2 \qquad (\bigstar)$$

Dist. recursion thus simplifies to

$$\mu_{t+1} = \sqrt{\lambda}\, \sigma_{t+1}^2 \qquad (1)$$

Note, $\mathbb{E}\left[\left(X - \mathbb{E}[X \mid \mu_t X + \sigma_t Z]\right)^2\right]$

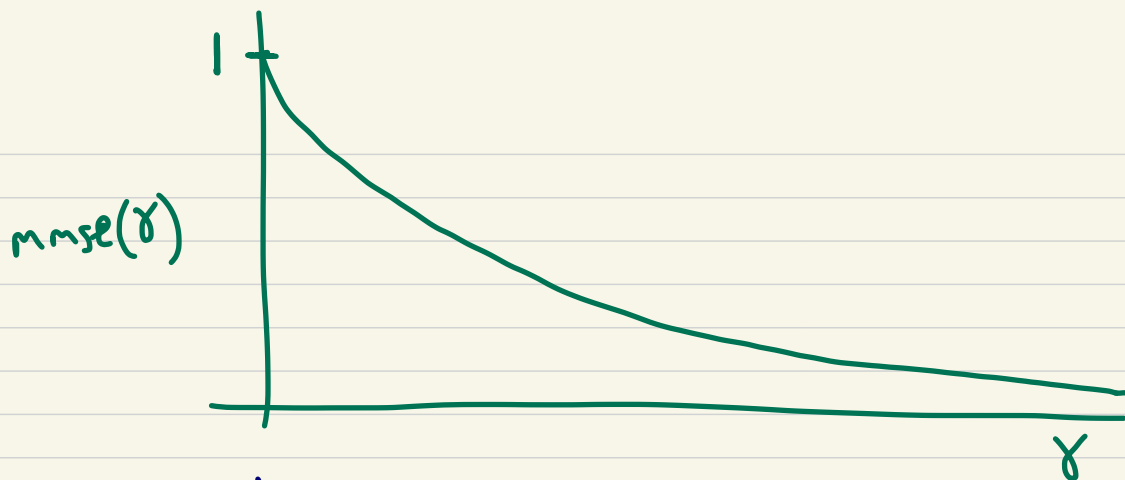$$= \underbrace{\mathbb{E}[X^2]}_{=1} - \mathbb{E}\left[\mathbb{E}[X \mid \mu_t X + \sigma_t Z]^2\right]$$

$$= 1 - \sigma_{t+1}^2 ,$$

this is MMSE for estimating X given $\mu_t X + \sigma_t Z$,

Define $\text{mmse}(\gamma)$ to MMSE for estimating X given

$\sqrt{\gamma}\, X + Z$. Then quantity $1-(\bigstar)$ above is

$$\text{mmse}\left(\frac{\mu_t^2}{\sigma_t^2}\right) = \text{mmse}\left(\lambda \sigma_t^2\right)$$

mmse(γ)

Other party dist. recursion thus simplifies to

$$\sigma^2_{t+1} = 1 - mmse\left(\lambda \sigma^2_t\right) \qquad (2)$$

Can combine (1) and (2) into 1D recursion by defining $\gamma_t = \lambda \sigma^2_t$. Then we get

$$\gamma_0 = 0, \qquad \boxed{\gamma_{t+1} = \lambda\left(1 - mmse(\gamma_t)\right)} \qquad (REC)$$

and $\mu_t = \frac{\gamma_t}{\sqrt{\lambda}}$, $\sigma^2_t = \frac{\gamma_t}{\lambda}$.

Can show that iterating (REC) will converge to fixed pt. $\gamma_\star^{(\lambda)}$ solving

$$\gamma_\star^{(\lambda)} = \lambda\left(1 - mmse(\gamma_\star^{(\lambda)})\right)$$