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Lecture 22: Approximate Message Passing

(See slides for derivation of AMP, state evolution)

If we set $f_t(y) = \mathbb{E}[X \mid \mu_t X + \sigma_t \xi = y]$,

then $\mathbb{E}_{X,\xi}[X f_t(\mu_t X + \sigma_t \xi)] = \mathbb{E}_{X,\xi}[f_t(\mu_t X + \sigma_t \xi)^2]$

$$= \mathbb{E}\left[\mathbb{E}[X \mid \mu_t X + \sigma_t \xi]^2\right] \qquad (\star)$$

Dist. recursion thus simplifies to

$$\mu_{t+1} = \sqrt{\lambda}\, \sigma_{t+1}^2 \qquad (1)$$

Note, $\mathbb{E}\left[\left(X - \mathbb{E}[X \mid \mu_t X + \sigma_t \xi]\right)^2\right]$

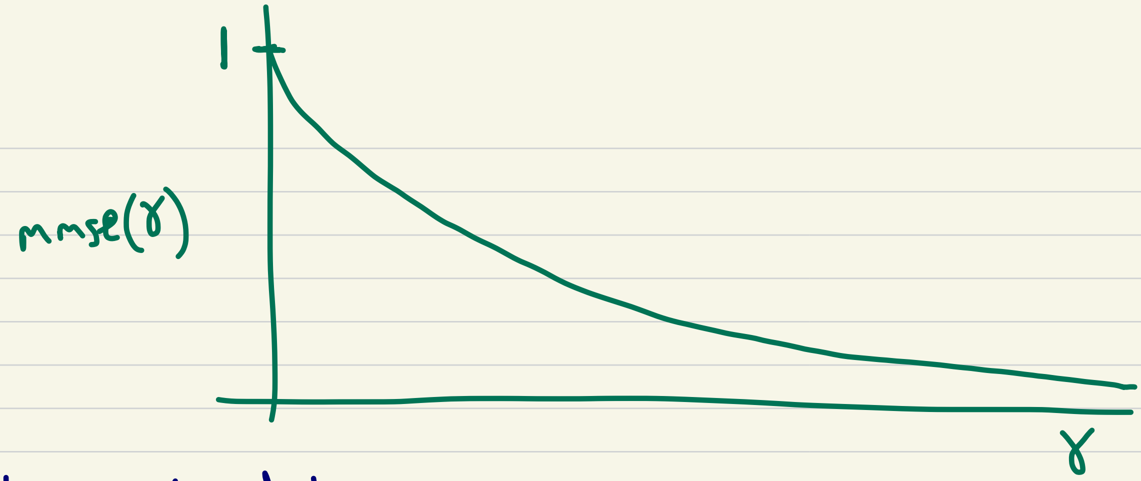
$$= \underbrace{\mathbb{E}[X^2]}_{=1} - \mathbb{E}\left[\mathbb{E}[X \mid \mu_t X + \sigma_t \xi]^2\right]$$

$$= 1 - (\star),$$

this is MMSE for estimating X given $\mu_t X + \sigma_t \xi$,

Define $\mathrm{mmse}(\gamma)$ to MMSE for estimating X given $\sqrt{\gamma} X + \xi$. Then quantity $1 - (\star)$ above is

$$\mathrm{mmse}\left(\frac{\mu_t^2}{\sigma_t^2}\right) = \mathrm{mmse}(\lambda \sigma_t^2)$$



Other part of dist. recursion thus simplifies to

$$\sigma_{t+1}^2 = 1 - \mathrm{mmse}\left(\lambda \sigma_t^2\right) \qquad (2)$$

Can combine (1) and (2) into 1D recursion by defining $\gamma_t = \lambda \sigma_t^2$. Then we get

$$\gamma_0 = 0, \quad \boxed{\gamma_{t+1} = \lambda\left(1 - \mathrm{mmse}(\gamma_t)\right)} \qquad (\mathrm{REC})$$

and $\mu_t = \frac{\gamma_t}{\sqrt{\lambda}}$, $\sigma_t^2 = \frac{\gamma_t}{\lambda}$.

Can show that iterating (REC) will converge to fixed pt. $\gamma_*(\lambda)$ solving

$$\gamma_*(\lambda) = \lambda\left(1 - \mathrm{mmse}\left(\gamma_*(\lambda)\right)\right)$$