

# Lecture 21: Belief propagation, Bethe free energy

Gibbs free energy on trees only depends on 1- and 2-wise marginals:

- Average energy:

$$E(x) = \sum_{(i,j) \in G} \lg(1/\psi_{ij}(x_i, x_j)), \text{ and}$$

$$\Phi(x) = \sum_{(i,j) \in G} \underbrace{\Phi \left( \lg 1/\psi_{ij}(x_i, x_j) \right)}_{\text{only depends on marginal dist on } (x_i, x_j)}$$

- Entropy: implied by the following:

Lemma: If underlying graph is a tree, then

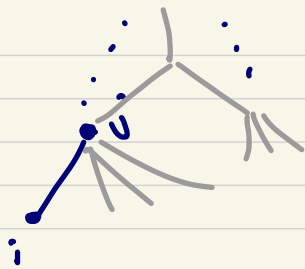
$$\mu(x) = \prod_{(i,j) \in G} \underbrace{\mu_{ij}(x_i, x_j)}_{\substack{\triangleq \text{marginal} \\ \text{dist. on} \\ (x_i, x_j)}} \prod_{i \in [n]} \underbrace{\mu_i(x_i)}_{\substack{\triangleq \text{marginal} \\ \text{dist. on} \\ x_i}} \quad \text{1-bit}$$

Pf: Induction on size of tree.

- Base case ( $n=1$ ) is vacuously true

- For inductive step, take any edge  $(i,j)$

connected to leaf, where  $i$  is leaf.



$$\begin{aligned}
 \Pr(x = s) &= \Pr(x_{n \setminus i} = s_{n \setminus i}) \cdot \Pr(x_i = s_i | x_{n \setminus i} = s_{n \setminus i}) \\
 &\stackrel{\text{(Markov property)}}{=} \Pr(x_{n \setminus i} = s_{n \setminus i}) \cdot \Pr(x_i = s_i | x_j = s_j) \\
 &= \underbrace{\Pr(x_{n \setminus i} = s_{n \setminus i})}_{M_{T'}(s_{n \setminus i})} \cdot \frac{M_{ij}(s_i, s_j)}{\mu_j(s_j)},
 \end{aligned}$$

for tree  $T'$  obtained by deleting  $i$

↑ contributes to  $(\mu_j)^{1 - |d_j|}$

Completing the inductive step.

Corollary:

$$H(\mu) = \sum_{(i,j) \in E} H(\mu_{ij}) - \sum_i (|d_i| - 1) H(\mu_i)$$

So entropy term in Gibbs free energy also only depends on 1- and 2-wise marginals.

Leads us to define Bethe free energy.

Domain of definition: marginals  $\{(v_i, v_{ij})\}$   
that satisfy "local consistency", i.e.

$$\sum_{x_j \in \{\pm 1\}} v_{ij}(x_i, x_j) = v_i(x_i) \quad \forall i, j, x_i$$

In CS, such marginals are said to satisfy  
"deg. 2 Sherali-Adams constraints" (LP version  
of pseudodistributions in SOS)

Bethe free energy:

$$G_B(\nu) \triangleq - \sum_{(ij) \in E} H(\nu_{ij}) + \sum_{i \in [n]} (|d_i| - 1) H(\nu_i) + \Phi_\nu(\xi)$$

For trees, we have seen that

Bethe free energy = Gibbs free energy

Claim 1. Fixed point of BP satisfies local consistency.

Pf: Recall BP gives marginals

$$V_i(\sigma) \propto \prod_{j \in \partial i} \bar{m}_{\sigma}^{j \rightarrow i}$$

$$V_{ij}(\sigma_i, \sigma_j) \propto \Psi_{ij}(\sigma_i, \sigma_j) m_{\sigma_i}^{i \rightarrow j} m_{\sigma_j}^{j \rightarrow i}$$

Note: for any  $x_i \in \{\pm 1\}$ ,

$$\sum_{x_j \in \{\pm 1\}} V_{ij}(x_i, x_j) = \sum_{x_j \in \{\pm 1\}} \Psi_{ij}(x_i, x_j) m_{x_i}^{i \rightarrow j} m_{x_j}^{j \rightarrow i}$$

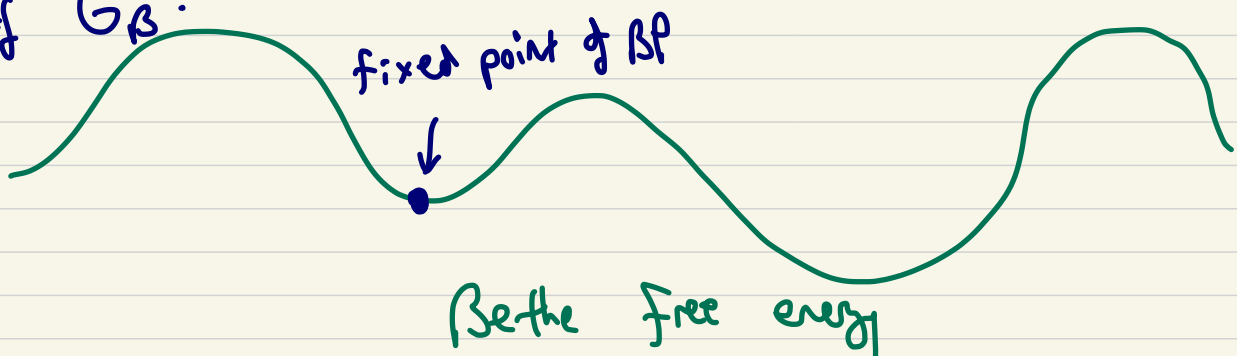
" $\bar{m}_{x_i}^{j \rightarrow i}$ " by  
fixed point assumption

$$\propto m_{x_i}^{i \rightarrow j} \bar{m}_{x_i}^{j \rightarrow i}$$

$$\propto V_i(x_i)$$

This holds for all  $x_i$ , so " $\propto$ " is actually " $=$ ".  $\square$

So BP finds some point in the optimization landscape of  $G_B$ :



Convenient formula for Bethe free energy:

Let

$$Z_i \triangleq \sum_{\sigma_i \in \{\pm 1\}} \prod_{j \in \partial_i} \bar{m}_{\sigma_i \rightarrow j}$$

$$Z_{ij} \triangleq \sum_{\sigma_i, \sigma_j \in \{\pm 1\}} \psi_{ij}(\sigma_i, \sigma_j) m_{\sigma_i \rightarrow j} \cdot m_{\sigma_j \rightarrow i}$$

$$Z_{ijj} \triangleq \sum_{\sigma_i \in \{\pm 1\}} m_{\sigma_i \rightarrow j} \bar{m}_{\sigma_i \rightarrow i}$$

Then

$$G_B[v] = - \sum_{i \in [n]} \ln Z_i - \sum_{(i,j) \in E} \ln Z_{ij} + \sum_{i \in [n]} \sum_{j \in \partial_i} \ln Z_{ijj}$$

Pf:

$$\begin{aligned}
 - \sum_{(i,j)} H(v_{ij}) + \sum_v \mathcal{E} &= - \sum_{(i,j)} \sum_{\sigma_i, \sigma_j} \ln \frac{\psi_{ij}(\sigma_i, \sigma_j)}{v_{ij}(\sigma_i, \sigma_j)} \\
 &= - \sum_{(i,j)} \sum_{\sigma_i, \sigma_j} \ln \frac{Z_{ij}}{m_{\sigma_i \rightarrow j} m_{\sigma_j \rightarrow i}} \\
 &= - \sum_{(i,j)} \ln Z_{ij} + \sum_{(i,j)} \sum_{\sigma_i, \sigma_j} \left[ \underbrace{m_{\sigma_i \rightarrow j}}_{+} + \underbrace{m_{\sigma_j \rightarrow i}}_{-} \right]
 \end{aligned}$$

$$-H(\nu_i) = - \sum_i \oplus_{\nu_i} \ln \frac{1}{\nu_i(x_i)}$$

$$= - \sum_i \oplus_{\nu_i} \ln \frac{Z_i}{\prod_{j \in d_i} \sum_{x_i} \nu_{ij}^{\otimes j}}$$

$$= - \sum_i \ln Z_i + \sum_i \oplus_{\nu_i} \left[ \sum_{j \in d_i} \ln \sum_{x_i} \nu_{ij}^{\otimes j} \right]$$

$$\sum_i |d_i| \cdot H(\nu_i) = \sum_i \sum_{j \in d_i} \oplus_{\nu_i} \ln \frac{1}{\nu_i(x_i)}$$

$$= \sum_i \sum_{j \in d_i} \oplus_{\nu_i} \ln \frac{Z_{ij}}{\sum_{x_i} \nu_{ij}^{\otimes j}}$$

$$= \sum_i \sum_{j \in d_i} \ln Z_{ij} - \sum_i \sum_{j \in d_i} \oplus_{\nu_i} \left[ \ln \sum_{x_i} \nu_{ij}^{\otimes j} + \ln \sum_{x_i} \nu_{ij}^{\otimes j} \right]$$

□

Thm: For any set of  $\{\psi_{ij}\}_{(ij) \in G}$ , even  
 if graph is not a tree, there is a 1-1  
 correspondence b/t:

fixed points of BP  $\longleftrightarrow$  Stationary points for  $G_\beta$  !

Pf:

$$\frac{\partial G_\beta(u)}{\partial m_{\sigma_i}^{\otimes \rightarrow j}} = \frac{1}{Z_{ij}} \frac{\partial}{\partial m_{\sigma_i}^{\otimes \rightarrow j}} Z_{ij} - \frac{1}{Z_{ij}} \frac{\partial}{\partial m_{\sigma_i}^{\otimes \rightarrow j}} Z_{ij}$$

$$= \frac{\sum_{\sigma_i} m_{\sigma_i}^{\otimes \rightarrow i}}{\sum_{\sigma_i} m_{\sigma_i}^{\otimes \rightarrow j} m_{\sigma_i}^{\otimes \rightarrow i}} - \frac{\sum_{\sigma_i} \psi_{ij}(\sigma_i, \sigma_j) m_{\sigma_i}^{\otimes \rightarrow i}}{\sum_{\sigma_i, \sigma_j} \psi_{ij}(\sigma_i, \sigma_j) m_{\sigma_i}^{\otimes \rightarrow j} m_{\sigma_j}^{\otimes \rightarrow i}}$$

This vanishing for all  $ij, \sigma_i$  is equivalent to:

$$m_{\sigma_i}^{\otimes \rightarrow i} \propto \sum_{\sigma_j} \psi_{ij}(\sigma_i, \sigma_j) m_{\sigma_j}^{\otimes \rightarrow i}$$

$$\frac{\partial G_\beta(u)}{\partial m_{\sigma_i}^{\otimes \rightarrow i}} = \frac{1}{Z_{ij}} \frac{\partial}{\partial m_{\sigma_i}^{\otimes \rightarrow i}} Z_{ij} - \frac{1}{Z_i} \frac{\partial}{\partial m_{\sigma_i}^{\otimes \rightarrow i}} Z_i$$

$$= \frac{\sum_{\sigma} \prod_{i \rightarrow j} m_{\sigma}^{i \rightarrow j}}{\sum_{\sigma} \prod_{k \in \mathcal{I}} \prod_{i \rightarrow k} m_{\sigma}^{i \rightarrow k}}$$

Again, this vanishing for all  $i, j, \sigma$  is equiv. to

$$m_{\sigma}^{i \rightarrow j} \propto \prod_{k \in \mathcal{I}} m_{\sigma}^{k \rightarrow i}$$

So stationarity  $\equiv$  BP fixed point.  $\square$