11/8/23

Lecture 18: Statistical Query Model (cont'd)

SQ dimension (recipe for supervised problems)

Def: Class of functions $\mathcal{F} = \{f : \mathbb{R} \to [-1,1]\}$ has SQ dimension $\geq D$ w.r.t q if $\exists\ f_1, \dots, f_D \in \mathcal{F}$ s.t. $\forall\ i \neq j$

$$\left| \mathop{\mathbb{E}}_{x \sim q} [f_i(x)\, f_j(x)] \right| \leq \frac{1}{D}.$$

Thm: If $\mathcal{F}$ has SQ dimension D, then CSQ learning requires tolerance τ or $\Omega(D\tau^2)$ queries

Pf: For convenience, define $\langle f, g \rangle \triangleq \mathbb{E}[f(x) g(x)]$.

WLOG let $\mathcal{F} = \{f_1, \dots, f_D\}$.

Given query $\phi : \mathbb{R}^d \to [-1, 1]$, let

$$A^+ \triangleq \{ f \in \mathcal{F} : \langle f, \phi \rangle \geq \tau \}$$

By Cauchy-Schwartz:

$$\left\langle \phi, \sum_{f \in A^+} f \right\rangle^2 \leq \underbrace{\|\phi\|^2}_{\leq 1} \cdot \left\| \sum_{f \in A^+} f \right\|^2$$

$$\leq \sum_{f \in A^+} \|f\|^2 + \frac{|A^+|(|A^+| - 1)}{D}$$

$$\leq \frac{|A^+|^2}{D} + |A^+|$$

But $\langle \phi, \sum_{f \in A^+} f \rangle \geq \tau |A^+|$ by defn., so

$$\tau^2 |A^+|^2 \leq \frac{|A^+|^2}{D} + |A^+|$$

$$\Rightarrow |A^+| \leq \frac{D}{D\tau^2 - 1} \leq O\left(\frac{1}{\tau^2}\right)$$

Similarly, for $A^- \triangleq \{ f \in \mathcal{F} : \langle f, \phi \rangle \leq -\tau \}$, $|A^-| \leq O\left(\frac{1}{\tau^2}\right)$.

So regardless of ϕ, all but $O\left(\frac{1}{\tau^2}\right)$ many f's consistent w/ the answer 0, so need $\Omega(D\tau^2)$ queries. □

SQ dimension bound for 1-hidden-layer MLPs:

[Goel-Gollakota-Jin-Karmalkar-Klivans '20]:

Let $S \subseteq [d]$ be of size $m = \lg_2 k$.

Given $w \in \{\pm 1\}^m$, define $w^{(S)} \in S^{d-1}$ by

$$w_i^{(S)} = \begin{cases} w_i/\sqrt{m} & \text{if } i \in S \\ 0 & \text{o.w} \end{cases}$$

Define $f_S : \mathbb{R}^d \to \mathbb{R}$ by

$$f_S(x) \triangleq \sum_{w \in \{\pm 1\}^m} \left(\prod_{i=1}^m w_i\right) \cdot \text{relu}\left(\langle w^{(S)}, x \rangle\right)$$

(Note: this is one-hidden-layer MLP w/ size $2^m = k$)

Claim: If q is sign-symmetric dist. over $\mathbb{R}$ (i.e. for any x, x and $-x$ are equally likely under q), then SQ dim of $\{f_S\}$ is $\geq \binom{d}{m}$.

Pf: For any distinct S, T of size m, w.t.s $\langle f_S, f_T \rangle_q = 0$

For any $z \in \{\pm 1\}^n$, note that

$$f_S(x \odot z) = \sum_w \prod_i w_i \cdot \underbrace{\mathrm{reln}(\langle w^{[S]}, x \odot z \rangle)}_{\mathrm{reln}(\langle w^{[S]} \odot z, x \rangle)}$$

$$= \sum_{w'} \prod_i w'_i \cdot z_S \cdot \mathrm{reln}(\langle w'^{[S]}, x \rangle)$$

$$= z_S \cdot f_S(x).$$

So $\langle f_S, f_T \rangle_q \overset{\text{by sign symmetry}}{=} \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_z \left[f_S(x \odot z) f_T(x \odot z) \right]$

$$= \mathop{\mathbb{E}}_x \mathop{\mathbb{E}}_z \left[f_S(x) f_T(x) z_S z_T \right]$$

$$= \mathop{\mathbb{E}}_x \left[f_S(x) f_T(x) \right] \cdot \underbrace{\mathop{\mathbb{E}}_z \left[z_S \cdot z_T \right]}_{\substack{\| \\ 0 \\ \text{if } S \neq T}} . \quad \square$$

NB: Technically, also have to make sure $\{f_S\}$ are nonzero functions! See paper for this calculation (Hermite analysis).

(Diakonikolas–Kane–Kontonis–Zarifis '20) showed stronger lbd ($d^{-\Omega(k)}$ queries or $2^{-\text{poly}(d)}$ tolerance)

using different instance of approximately orthogonal functions: given 2D subspace $U \subseteq \mathbb{R}^d$,

$$f_U(x) = h(\langle u_1, x\rangle, \langle u_2, x\rangle)$$

where $$h(a,b) = \sum_{i=1}^{k} (-1)^i \left\langle \left(\cos\left(\tfrac{2\pi}{k}\right), \sin\left(\tfrac{2\pi}{k}\right)\right), (a,b)\right\rangle.$$

and u_1, u_2 are basis for U.

idea: $\deg \le \frac{k+1}{2}$ Hermite coeffs of f_U are 0, so

$$\langle f_U, f_V\rangle_{N(0,I_d)} = \langle f_U^{>\frac{k+1}{2}}, f_V^{>\frac{k+1}{2}}\rangle$$

$$\leq \left(\text{"correlation}(U,V)\text{"} \right)^{\ell(k)}.$$

So if we take a packing of the space of 2D subspaces, we get the desired lbd on the SQ dimension.

Statistical dimension:

Def: Set of distributions T has **statistical dimension** $\geq \Delta$ w.r.t. reference distribution D and avg. correlation γ if $\forall$ T of size $\geq |T|/\Delta$,

$$\rho_D(T) \triangleq \frac{1}{|T|^2} \sum_{D_i, D_j \in T} \langle D_i, D_j \rangle_D \leq \gamma,$$

where $$\langle D_i, D_j \rangle_D = \mathop{\mathbb{E}}_{x \sim D} \left[\left(\frac{D_i(x)}{D(x)} - 1 \right) \left(\frac{D_j(x)}{D(x)} - 1 \right) \right]$$

Thm: If T^* has statistical dimension $\geq \Delta$, then learning dist's in T^* w/ SQ alg. requires either $\Omega(\Delta)$ queries or $\sqrt{\gamma}$ tolerance.

Pf: Given query $\phi : \mathbb{R}^d \to \mathbb{R}$, define

$$A^+ \triangleq \left\{ D' \in T^\delta \text{ s.t. } \mathop{\mathbb{E}}_{D'}[\phi] \geq \mathop{\mathbb{E}}_{D}[\phi] + \sqrt{\gamma} \right\}$$

$$A^- \triangleq \left\{ D' \in T^\delta \text{ s.t. } \mathop{\mathbb{E}}_{D'}[\phi] \leq \mathop{\mathbb{E}}_{D}[\phi] - \sqrt{\gamma} \right\},$$

i.e. set of dist's in T^δ s.t. answering ϕ w/ $\mathop{\mathbb{E}}_{D}[\phi]$ is incorrect. w.r.t.s $|A^+|$, $|A^-|$ small.

$$\mathop{\mathbb{E}}_{x \sim D}\left[\phi(x) \cdot \sum_{D' \in A^+}\left\{\frac{D'(x)}{D(x)} - 1\right\}\right]^2 \leq \underbrace{\mathop{\mathbb{E}}_{x}\left[\phi(x)^2\right]}_{\leq 1} \cdot \mathop{\mathbb{E}}_{x}\left[\left(\sum_{D' \in A^+}\left[\frac{D'(x)}{D(x)} - 1\right]\right)^2\right]$$

$$= \sum_{D', D'' \in A^+} \langle D', D'' \rangle_D$$

$$\leq |A^+|^2 \cdot \beta_D(A^+)$$

Remains to lower bound

$$\mathop{\mathbb{E}}_{x \sim D}\left[\phi(x) \cdot \sum_{D' \in A^+}\left\{\frac{D'(x)}{D(x)} - 1\right\}\right]^2. \qquad (b)$$

Note: $\mathop{\mathbb{E}}_{x \sim D}\left[\phi(x) \cdot \left\{\frac{D'(x)}{D(x)} - 1\right\}\right] = \mathop{\mathbb{E}}_{x \sim D'}[\phi(x)] - \mathop{\mathbb{E}}_{x \sim D}[\phi(x)]$

$\geq \sqrt{\gamma}$,

So $(\star) \geq |A^+|^2 \cdot \gamma$, so

$$\rho_D(A^+) \geq \gamma,$$

implying $|A^+| / |T^*| \leq 1/\Delta$.

Similarly, $|A^-| / |T^*| \leq 1/\Delta$.

So need $\Omega(\Delta)$ queries b/c each one only rules out a $1/\Delta$ fraction. $\square$

(Bonus material)

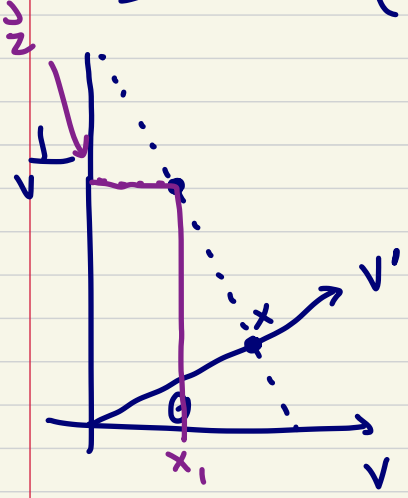
Moment-matching and SQ:

Consider P_v and $P_{v'}$ (defined in slides) for moment-matching distribution A. WLOG suppose $v = (1, 0, \ldots, 0)$, $v' = (\cos\theta, \sin\theta, 0, \ldots, 0)$.

Claim: Let Q be dist over $\mathbb{R}^d$ given by taking $X \sim P_v$ and outputting $v' \cdot x$.

Then $\langle Q, Q \rangle_{N(0,1)} \leq (\cos\theta)^{2(n+1)} \langle A, A \rangle_{N(0,1)}$

Pf: $Q(x) = \int_{\vec{x} \cdot (\vec{v}, v') = x} P_V(\vec{x})\, d\vec{x}$



$= \int_{(x_1, \vec{z}):\ \langle (x_1, \vec{z}), v' \rangle = x} A(x_1) \cdot G(\vec{z})\, dx_1 d\vec{z}$

Gaussian

$\langle (x_1, \vec{z}), v' \rangle = x \iff x_1 \cos\theta + z_1 \sin\theta = x$

Scaled convolution of A and Gaussian

$= \int_{(x_1, z_1):\ x_1 \cos\theta + z_1 \sin\theta = x} A(x_1)\, G(z_1)\, dx_1 dz_1$

$= \mathbb{E}_{g \sim N(0,1)} \left[A(\cos\theta\, x + \sin\theta\, g) \right]$

$\leq \underbrace{\bigcup_{\theta} A(x)}$

"Ornstein-Uhlenbeck noise operator"

Write $\frac{A(x)}{G(x)} = \sum_{i=0}^{\infty} a_i \underbrace{He_i(x)}_{\text{probabilist's Hermite polynomial}} \cdot \frac{1}{\sqrt{i!}}$

note: $\forall i \le m$ $a_i = \mathbb{E}_{x \sim N(0,I)}\left[\frac{A(x)}{G(x)} He_i(x) \cdot \frac{1}{\sqrt{i!}}\right]$

$$= \int A(x)\, He_i(x) \cdot \frac{1}{\sqrt{i!}}\, dx$$

$$\overset{\text{b/c moment matching}}{=} \int G(x)\, He_i(x) \cdot \frac{1}{\sqrt{i!}}\, dx \overset{\text{orthogonality}}{=} 0 \qquad (\star\star)$$

Mehler's lemma: $U_\theta (He_i \cdot G)(x) =$

$$\cos^i(\theta)\, He_i(x)\, G(x),$$

Intuition: noise "smears out" high-degree info

i.e. (Hermite × Gaussian density) is eigenfunction of noise operator

So $U_\theta A(x) = \sum_{i=0}^{\infty} a_i \cos^i(\theta)\, He_i(x)\, G(x) \frac{1}{\sqrt{i!}}$

$$\overset{\text{(by } (\star\star))}{=} \sum_{i=m+1}^{\infty} a_i \cos^i(\theta)\, He_i(x)\, G(x) \frac{1}{\sqrt{i!}}$$

$$\langle Q, Q \rangle_{N(0,1)} = \langle U_\theta A, U_\theta A \rangle_{N(0,1)}$$

$$= \sum_{i=m+1}^{\infty} q_i^2 \cos^{2i}(\theta)$$

$$\leq \cos^{2m+2}(\theta) \cdot \underbrace{\sum_{i=m+1}^{\infty} q_i^2}_{\leq 1} \langle A, A \rangle_{N(0,1)}. \quad \square$$

We conclude that

$$\langle P_V, P_{V'} \rangle_{N(0,Id)} = \langle A, U_\theta A \rangle_{N(0,1)}$$

by the above

$$\leq \langle A, A \rangle_{N(0,1)}^{1/2} \cdot \langle U_\theta A, U_\theta A \rangle_{N(0,1)}^{1/2}$$

$$\leq \underbrace{\cos^{m+1}(\theta)}_{\text{exp. decaying in } m} \langle A, A \rangle_{N(0,1)}. \quad \square$$